

APPROXIMATION PROPERTIES AND ENTROPY ESTIMATES FOR CROSSED PRODUCTS BY ACTIONS OF AMENABLE DISCRETE QUANTUM GROUPS

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ABSTRACT. We construct explicit approximating nets for crossed products of C^* -algebras by actions of discrete quantum groups. This implies that C^* -algebraic approximation properties such as nuclearity, exactness or completely bounded approximation property are preserved by taking crossed products by actions of amenable discrete quantum groups. We also show that the noncommutative topological entropy of a transformation commuting with the quantum group action does not change when we pass to the canonical extension to the crossed product. Both these results are extended to twisted crossed products via a stabilisation trick.

Studying various finite-dimensional approximation properties such as nuclearity or exactness has become in recent years one of the central areas of investigations in the theory of C^* -algebras. We refer to the book [BO] for a state-of-the-art treatment of the subject. One of the natural questions is whether standard constructions of C^* -algebras preserve approximation properties. As there exist strong connections and analogies between the theory of approximations in operator algebras and amenability of groups, it is natural to expect that $B \rtimes_\alpha G$, a crossed product of a C^* -algebra B by an action α of an amenable group G should have the same approximation properties as B . This is indeed the case, as one can construct explicit factorisations of $B \rtimes_\alpha G$ through finite matrices over B ([Vo], [SS], see also Chapter 4.2 of [BO]). These factorisations are of Schur multiplier type and the fact that one can construct a net of such factorisations pointwise convergent to the identity map on $B \rtimes_\alpha G$ follows from the existence of a family of ‘approximately invariant’ finitely supported functions on an amenable group.

In this paper we show the existence of analogous factorisations for crossed products of C^* -algebras by actions of amenable discrete *quantum* groups ([Ku₂], [To]). As a discrete quantum group A is a noncommutative C^* -algebra in general, it does not make sense to speak directly about finitely supported functions on such an object. On the other hand there is a natural notion of ‘finitely supported’ vectors in H_φ , the Hilbert space arising from the GNS construction applied to the left invariant weight on A . Recently R. Tomatsu showed in [To] that amenability of a discrete quantum group is equivalent to the existence of a net of finitely supported vectors in H_φ which are approximately invariant in the appropriate sense (see Theorem 4.3 below). Exploiting this fact together with the explicit construction of factorisations

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allows us to show that if A is an amenable discrete quantum group acting on a C^* -algebra B , then the reduced crossed product $\hat{A} \rtimes_{\alpha} B$ is nuclear (respectively is exact, has OAP, CBAP or strong OAP) if and only if B is nuclear (respectively is exact, has OAP, CBAP or strong OAP). In [VVe] S. Vaes and R. Vergnioux showed that if A is amenable then the reduced crossed product $\hat{A} \rtimes_{\alpha} B$ coincides with the universal one and applied this to obtain the above result for nuclearity. Analogous results for exactness of crossed products by the actions of amenable Hopf C^* -algebras and amenable multiplicative unitaries can also be found in [Ng] and [BS]. The advantage of our method lies in providing explicit approximations, which are further used to show that if B is unital and nuclear and γ is a unital completely positive map commuting with an action α of an amenable discrete quantum group A , then the Voiculescu topological entropy of γ coincides with the entropy of the canonical extension of γ to $\hat{A} \rtimes_{\alpha} B$.

The plan of the paper is as follows: after introducing basic notations we proceed in Section 1 to recall basic definitions and statements related to the theory of locally compact quantum groups of J. Kustermans and S. Vaes, with the special emphasis put on discrete quantum groups. In Section 2 we recall the notion of an action of a locally compact quantum group on a C^* -algebra and its corresponding reduced/universal crossed products. Section 3 contains the explicit construction of factorisations and Section 4 the application of these to the main results of the paper, together with the characterisation of amenable discrete quantum groups due to R. Tomatsu. Finally, in Section 5 we adapt the von Neumann algebraic stabilisation trick of L. Vainerman and S. Vaes to the C^* -algebraic framework to show that our results remain valid for the crossed products given by twisted (cocycle) actions of amenable discrete quantum groups.

GENERAL NOTATIONS

All inner products in this paper are conjugate linear in the *first* variable. For a pair of vectors ξ, η in a Hilbert space K the normal functional $\omega_{\xi, \eta} \in B(K)_*$ is given by the formula

$$\omega_{\xi, \eta}(T) = \langle \xi, T\eta \rangle, \quad T \in B(K).$$

We will also use the Dirac-type notation $\langle \xi |$ and $|\eta \rangle$ for obvious operators in $B(K; \mathbb{C})$ and $B(\mathbb{C}; K)$ respectively. Note that if K' is an additional Hilbert space and $S \in B(K \otimes K')$ then

$$(\omega_{\xi, \eta} \otimes \text{id}_{B(K')})(S) = (\langle \xi | \otimes I_{K'})S(|\eta \rangle \otimes I_{K'})$$

and

$$((\omega \otimes \text{id}_{B(K')})(S))^* = (\omega^* \otimes \text{id}_{B(K')})(S^*).$$

If $a \in B(H)$ we use the standard notation $\omega a, a\omega$ for normal functionals on $B(H)$ given by

$$(\omega a)(T) = \omega(aT), \quad (a\omega)(T) = \omega(Ta), \quad T \in B(H).$$

The symbol \otimes will always signify the minimal or spatial tensor product of C^* -algebras, $\bar{\otimes}$ the ultraweak tensor product of $(\sigma$ -weakly continuous maps on) von Neumann algebras, whereas \odot denotes the algebraic tensor product. $F \subset\subset G$ means that F is a finite subset of G .

1. LOCALLY COMPACT QUANTUM GROUPS - BASIC NOTATIONS AND DEFINITIONS

The concept of locally compact quantum groups was introduced by J. Kustermans and S. Vaes in [KV]. A detailed description of the motivation and general development of the theory can be found in [Ku₂]; we follow the notation used in [To].

Multiplier algebras. The multiplier algebra $\mathcal{M}(\mathbb{C})$ of a C^* -algebra \mathbb{C} is the largest C^* -algebra in which \mathbb{C} sits as an essential ideal. As we often work with tensor products of C^* -algebras we need to describe the algebras of ‘one-legged’ multipliers.

Definition 1.1. Let \mathbb{B}, \mathbb{C} be C^* -algebras. The \mathbb{B} -multiplier algebra of $\mathbb{B} \otimes \mathbb{C}$ is

$$\mathcal{M}_l(\mathbb{B} \otimes \mathbb{C}) = \{d \in \mathcal{M}(\mathbb{B} \otimes \mathbb{C}) : d(b \otimes 1), (b \otimes 1)d \in \mathbb{B} \otimes \mathbb{C} \text{ for all } b \in \mathbb{B}\}.$$

Similarly the \mathbb{C} -multiplier algebra of $\mathbb{B} \otimes \mathbb{C}$ is

$$\mathcal{M}_r(\mathbb{B} \otimes \mathbb{C}) = \{d \in \mathcal{M}(\mathbb{B} \otimes \mathbb{C}) : d(1 \otimes c), (1 \otimes c)d \in \mathbb{B} \otimes \mathbb{C} \text{ for all } c \in \mathbb{C}\}.$$

It is easy to see that both multiplier algebras defined above are C^* -subalgebras of $\mathcal{M}(\mathbb{B} \otimes \mathbb{C})$, with $\mathcal{M}_l(\mathbb{B} \otimes \mathbb{C})$ unital if and only if \mathbb{C} is unital.

For a careful discussion of ‘one-legged’ multiplier algebras, their natural topologies and extensions of maps defined on the algebraic tensor product we refer to Section 1 of Appendix A of [EKQR]. Here note only that the use of multipliers is unavoidable when we want to discuss actions of locally compact (quantum) groups on non-unital C^* -algebras (c.f. the discussion after Definition 2.1).

If \mathbb{C} is a direct (c_0 -type) sum of matrix algebras, $\mathbb{C} = \bigoplus_{\beta \in \mathcal{J}} M_{n_\beta}$, then $\mathcal{M}_l(\mathbb{B} \otimes \mathbb{C}) \approx \prod_{\beta \in \mathcal{I}} M_{n_\beta}(\mathbb{B})$. This is relevant for the later discussion of discrete quantum groups.

Locally compact quantum groups - von Neumann algebraic setting.

Definition 1.2. A pair (\mathbb{M}, Δ) is called a locally compact quantum group (in the von Neumann algebraic setting) if \mathbb{M} is a von Neumann algebra, $\Delta : \mathbb{M} \rightarrow \mathbb{M} \otimes \mathbb{M}$ is a normal unital $*$ -homomorphism satisfying the coassociativity property

$$(\Delta \overline{\otimes} \text{id}_{\mathbb{M}})\Delta = (\text{id}_{\mathbb{M}} \overline{\otimes} \Delta)\Delta$$

and there exist normal semifinite faithful left and right invariant weights φ and ψ on \mathbb{M} .

For the appropriate definition of left and right invariance we refer to [Ku₂]. We will always consider \mathbb{M} in its canonical representation on the GNS-space of the weight φ , further denoted by \mathbb{H}_φ . One can associate to the pair (\mathbb{M}, Δ) the *multiplicative unitary* $W \in B(\mathbb{H}_\varphi \otimes \mathbb{H}_\varphi)$ ([BS]). It contains all the information about the locally compact quantum group (\mathbb{M}, Δ) ; in particular

$$\Delta(m) = W^*(I_{\mathbb{H}_\varphi} \otimes m)W, \quad m \in \mathbb{M}.$$

Define for each $\omega \in B(\mathbb{H}_\varphi)_*$

$$\lambda(\omega) = (\omega \otimes \text{id}_{\mathbb{H}_\varphi})(W) \in B(\mathbb{H}_\varphi)$$

and let

$$\hat{\mathcal{A}} = \{\lambda(\omega) : \omega \in B(\mathbb{H}_\varphi)_*\}.$$

The *dual locally compact quantum group (in the von Neumann algebraic setting)* $\hat{\mathbb{M}}$ is defined as the σ -weak closure of $\hat{\mathcal{A}}$. The coproduct on $\hat{\mathbb{M}}$ is defined via the

multiplicative unitary $\hat{W} = \Sigma W^* \Sigma$, where Σ is the unitary implementing the tensor flip on $H_\varphi \otimes H_\varphi$. More precisely, $\hat{\Delta}$ is defined by the formula

$$\hat{\Delta}(x) = \hat{W}^*(I_{H_\varphi} \otimes x)\hat{W}, \quad x \in \hat{M}.$$

Both M and \hat{M} are in standard form on H_φ (so that in particular all normal states on M and \hat{M} can be realised on H_φ as vector states).

For any $\omega \in B(H_\varphi)_*$ define the (right) convolution operator on \hat{M} by

$$(1.1) \quad T_\omega(x) = (\text{id}_{B(H_\varphi)} \otimes \omega)\hat{\Delta} : \hat{M} \rightarrow \hat{M}.$$

If ω is a state then T_ω is unital and completely positive.

Locally compact quantum groups - C^* -algebraic setting. Let (M, Δ) be a locally compact quantum group in the von Neumann algebraic setting and let $W \in B(H_\varphi \otimes H_\varphi)$ be the associated multiplicative unitary. Define

$$\mathcal{A} = \{(\text{id}_{B(H_\varphi)} \otimes \omega)(W) : \omega \in B(H_\varphi)_*\}.$$

Let A denote the norm closure of \mathcal{A} . It turns out to be a C^* -subalgebra of M ([Ku₂]), the coproduct $\Delta|_A$ takes values in the multiplier algebra of $A \otimes A$ and the pair $(A, \Delta|_A)$ is called a locally compact quantum group in the C^* -algebraic setting associated to (M, Δ) . We will often denote it simply by (A, Δ) .

The (*reduced*) *dual locally compact quantum group (in the C^* -algebraic setting)* \hat{A} is given by the norm closure of $\hat{\mathcal{A}}$. Again the dual comultiplication $\hat{\Delta}$ on \hat{M} restricts to a map from \hat{A} to $\mathcal{M}(\hat{A} \otimes \hat{A})$. Moreover

$$W \in \mathcal{M}(A \otimes \hat{A}).$$

The right convolution operators defined in (1.1) yield by restriction maps from \hat{A} to $\mathcal{M}(\hat{A})$.

It is also possible to give an intrinsic definition of a locally compact quantum group in the C^* -algebraic setting ([Ku₂]) or to consider a universal, representation independent approach ([Ku₁]).

Further we will mainly work with the C^* -algebraic locally compact quantum groups, always represented on the Hilbert space given by the Haar weight φ . Thus the notations $A, \hat{A}, \varphi, W, \hat{W}, \Delta, H_\varphi$ will be subsequently used without any additional comments, viewing A and \hat{A} as subalgebras of $B(H_\varphi)$.

Discrete quantum groups. A locally compact quantum group A is called *discrete* if \hat{A} is unital (in other words, \hat{A} is a *compact quantum group*). Any discrete quantum group possesses a canonical one-dimensional central projection z_ϵ giving rise to a *counit*, i.e. a character $\epsilon \in A^*$ such that

$$(\epsilon \otimes \text{id}_A)\Delta = (\text{id}_A \otimes \epsilon)\Delta = \text{id}_A.$$

The counit extends uniquely to a normal character on M again satisfying the obvious modification of the above property.

Furthermore, if A is a discrete quantum group then there exists a family of central projections $(z_i)_{i \in \mathcal{I}}$ such that

$$A = \bigoplus_{i \in \mathcal{I}} Az_i,$$

and for each $i \in \mathcal{I}$ there exists $n_i \in \mathbb{N}$ such that $Az_i \approx M_{n_i}$. Moreover the multiplicity of the inclusion of Az_i into $B(H_\varphi)$ is equal to n_i , so that $(z_i)_{i \in \mathcal{I}}$ can be

viewed as a family of mutually orthogonal finite-dimensional projections in $B(H_\varphi)$ summing to $1_{B(H_\varphi)}$ (A is represented on H_φ nondegenerately). If F is a finite subset of \mathcal{I} we write $z_F = \sum_{i \in F} z_i$. A vector $\xi \in H_\varphi$ is said to be *finitely supported* if there exists a finite set $F \subset \mathcal{I}$ such that $\xi \in z_F H_\varphi$.

All the above statements can be found for example in [Ku₂] and can be regarded as a natural extension of the Peter-Weyl theory. As discrete quantum groups are duals of compact quantum groups, they can be thought of as encoding the (co)representation theory of a given compact quantum group. It is also possible to define C^* -algebraic discrete quantum groups directly, without referring to the duality (Definition 3.19 in [Ku₂]).

Note that the fact that $W \in \mathcal{M}(A \otimes \hat{A})$ implies that for all $i \in \mathcal{I}$

$$W(z_i \otimes I_{H_\varphi}) = (z_i \otimes I_{H_\varphi})W.$$

2. THE NOTION OF A CROSSED PRODUCT BY AN ACTION OF A QUANTUM GROUP

This section contains a general discussion of crossed products of C^* -algebras by actions of locally compact quantum groups. Although none of the concepts introduced below is new, in the existing literature they are usually discussed in the von Neumann algebraic context ([Va₁], [VVa]) or with the locally compact quantum groups replaced by C^* -Hopf algebras ([Ng]) or *weak Kac systems* ([BS], [Ti]).

As we are mainly interested in the ‘reduced’ framework, the actions we consider will take values in the minimal tensor product. The universal theory requires dealing with many technical subtleties, even when *coactions* of groups are considered (see [EKQR]).

Definition 2.1. A (left) action of a locally compact quantum group A on a (unital) C^* -algebra B is a nondegenerate (unital) $*$ -homomorphism $\alpha : B \rightarrow \mathcal{M}(A \otimes B)$ such that

$$(2.1) \quad (\Delta \otimes \text{id}_B) \circ \alpha = (\text{id}_A \otimes \alpha) \circ \alpha.$$

The left action α is said to be nondegenerate (or continuous in the strong sense) if $\alpha : B \rightarrow \mathcal{M}_1(A \otimes B)$ and $\overline{\alpha(B)(A \otimes 1_{M(B)})} = A \otimes B$.

There is an analogous concept of a right action ($\alpha : B \rightarrow \mathcal{M}_r(B \otimes A)$). As we are only interested in the case where A is a discrete quantum group, all the actions we consider in this paper are **left nondegenerate actions**, and the specification will be omitted in the sequel. For a discussion of various notions of continuity for actions of general locally compact quantum groups we refer to [BSV].

Classically by an action of a locally compact group G on a C^* -algebra B is meant a homomorphism $\tilde{\alpha} : G \rightarrow \text{Aut}(B)$ which is pointwise-norm continuous, i.e. for each $b \in B$ the function $g \mapsto \tilde{\alpha}_g(b)$ is continuous. Given $\tilde{\alpha}$ as above define $\alpha : B \rightarrow \mathcal{M}_1(C_0(G) \otimes B) = C_b(G; B)$ by

$$\alpha(b)(g) = \tilde{\alpha}_g(b), \quad g \in G, b \in B.$$

It is easy to see that α is then an action of $C_0(G)$ on B according to Definition 2.1 – recall that the coproduct on $C_0(G)$ is given by the formula $\Delta(f)(g, h) = f(gh)$ ($f \in C_0(G), g, h \in G$). Conversely, if α is an action of $C_0(G)$ on B then we can define for each $g \in G$ an automorphism $\tilde{\alpha}_g$ by

$$\tilde{\alpha}_g(b) = \alpha(b)(g), \quad b \in B,$$

and the resulting map $\tilde{\alpha} : G \rightarrow \text{Aut}(\mathbf{B})$ is a point-norm continuous homomorphism. Thus classical actions of a group G are in 1-1 correspondence with (left nondegenerate) actions of the locally compact quantum group $C_0(G)$. As $\mathcal{M}(\mathbf{B} \otimes C_0(G)) = C_b^{\text{strict}}(G; \mathcal{M}(\mathbf{B}))$ (where ‘strict’ refers to functions continuous in the strict topology on $\mathcal{M}(\mathbf{B})$), we see why it is important to consider ‘one-legged’ multiplier algebras.

The (reduced) coactions of a group G , as considered for example in [EKQR], correspond exactly to actions of the locally compact quantum group $C_r^*(G)$.

If \mathbf{A} is a discrete quantum group, then $\mathbf{A} = \bigoplus_{i \in \mathcal{I}} M_{n_i}$ and the action of \mathbf{A} on a C^* -algebra \mathbf{B} is given by a family $(\alpha_i)_{i \in \mathcal{I}}$ of nondegenerate $*$ -homomorphisms from \mathbf{B} to $M_{n_i}(\mathbf{B})$, satisfying extra requirements given by the condition (2.1). If \mathbf{A} is the dual of a compact group G , then the condition (2.1) describes a certain covariance property with respect to the fusion rules of representations of G (as the latter are encoded by the formula for the coproduct with respect to the identification of $\widehat{C(G)}$ with a direct sum of matrices).

We will need the following lemma clarifying the connections between various conditions expressing nondegeneracy/faithfulness/continuity of actions of discrete quantum groups.

Lemma 2.2. *Let \mathbf{A} be a discrete quantum group and let \mathbf{B} be a C^* -algebra. Assume that $\alpha : \mathbf{B} \rightarrow \mathcal{M}_l(\mathbf{A} \otimes \mathbf{B})$ is a nondegenerate $*$ -homomorphism satisfying the ‘action equation’ (2.1). The following conditions are equivalent:*

- (i) $\overline{\alpha(\mathbf{B})(\mathbf{A} \otimes 1_{M(\mathbf{B})})} = \mathbf{A} \otimes \mathbf{B}$;
- (ii)
- (2.2) $(\epsilon \otimes \text{id}_{\mathbf{B}}) \circ \alpha = \text{id}_{\mathbf{B}}$;
- (iii) α is faithful.

Proof. Define $\alpha_0 = (\epsilon \otimes \text{id}_{\mathbf{B}}) \circ \alpha$. Applying $\text{id}_{\mathbf{A}} \otimes \epsilon \otimes \text{id}_{\mathbf{B}}$ to (2.1) yields

$$(\text{id}_{\mathbf{B}} \otimes \alpha_0) \circ \alpha = \alpha,$$

so that for any $a \in \mathbf{A}$, $b \in \mathbf{B}$

$$\alpha(b)(a \otimes 1_{M(\mathbf{B})}) = (\text{id}_{\mathbf{B}} \otimes \alpha_0)(\alpha(b))(a \otimes 1_{M(\mathbf{B})}) = (\text{id}_{\mathbf{B}} \otimes \alpha_0)(\alpha(b)(a \otimes 1_{M(\mathbf{B})})).$$

This means that (i) implies (ii). The implication (ii) \implies (iii) is clear; so is its converse, as applying $\epsilon \otimes \text{id}_{\mathbf{A}} \otimes \text{id}_{\mathbf{B}}$ to equation (2.1) yields $\alpha = \alpha \circ \alpha_0$. Note that all these do not use the fact that \mathbf{A} is discrete and remain valid for any *coamenable locally compact quantum group* (i.e. a locally compact quantum group for which \mathbf{A} admits a bounded counit).

Assume then that (ii) holds. Recall that $\mathbf{A} = \mathbb{C} \oplus \bigoplus_{i \in \mathcal{I}} \mathbf{A}_i$, where for each $i \in \mathcal{I}$ there exists $n_i \in \mathbb{N}$ such that $\mathbf{A}_i = M_{n_i}$. The resulting canonical central projections in \mathbf{A} will be denoted by z_i ; the counit ϵ is given by the scalar in the first factor in the direct sum above. As stated before the lemma one can decompose α into a direct sum of nondegenerate $*$ -homomorphisms $\alpha_i : \mathbf{B} \rightarrow M_{n_i}(\mathbf{B})$. We need to prove that for each $i \in \mathcal{I}$ the coefficient space of $\alpha_i(\mathbf{B})$ in $M_{n_i}(\mathbf{B})$ given by $C_i = \overline{\text{Lin}\{(\omega_i \otimes \text{id}_{\mathbf{B}})(\alpha_i(b)) : \omega_i \in M_{n_i}^*, b \in \mathbf{B}\}}$ is equal to \mathbf{B} . Recall that the algebraic direct sum of \mathbb{C} and the matrix algebras \mathbf{A}_i is an algebraic quantum group in the sense of van Daele ([vD₁], [vD₂]), denoted further by \mathcal{A} . In particular there exists an antipode, a linear map $S : \mathcal{A} \rightarrow \mathcal{M}(\mathcal{A})$ (where now the multiplier algebra is also understood in the purely algebraic sense) such that

$$m((S \otimes \text{id}_{\mathcal{A}})(\Delta(a_1)(a_2 \otimes 1))) = \epsilon(a_1)a_2$$

for all $a_1, a_2 \in \mathcal{A}$. (m denotes the multiplication from $\mathcal{M}(\mathcal{A}) \odot \mathcal{A}$ to \mathcal{A} .) Moreover, for each $i \in \mathcal{I}$ there exists $\bar{i} \in \mathcal{I}$ such that $S(z_i) = z_{\bar{i}}, S(A_i) = A_{\bar{i}}$. We know also that for each $i, j \in \mathcal{I}$ there exist only finitely many $k \in \mathcal{I}$ such that $(z_i \otimes z_j)\Delta(z_k) \neq 0$. Now let $i \in \mathcal{I}$ and $b \in B$. Then $D := (z_{\bar{i}} \otimes z_i \otimes 1_{\mathcal{M}(B)})(\Delta \otimes \text{id}_A)(\alpha(b)) \in A_{\bar{i}} \otimes A_i \otimes B$, so that we can apply to the above $S \otimes \text{id}_A \otimes \text{id}_B$ with the result in $\mathcal{A} \odot \mathcal{A} \odot B$. Further we can view D as the following (finite!) sum:

$$D = \sum_{k \in \mathcal{I}} (z_{\bar{i}} \otimes z_i \otimes 1_{\mathcal{M}(B)})(\Delta \otimes \text{id}_A)(\alpha_k(b))(z_k \otimes 1_{M(B)}),$$

so that using the antimultiplicative property of S we obtain

$$\begin{aligned} & (m_{12} \odot \text{id}_B)(S \odot \text{id}_A \odot \text{id}_B)(D) \\ &= \sum_{k \in \mathcal{I}} (m_{12} \odot \text{id}_B)(z_i \otimes 1_{M(A)} \otimes 1_{M(B)})((S \odot \text{id}_A)(\Delta(z_k \alpha_k(b)_{(1)})(1_{M(A)} \otimes z_i))) \otimes \alpha_k(b)_{(2)}) \\ &= \sum_{k \in \mathcal{I}} \epsilon(z_k \alpha_k(b)_{(1)}) z_i \otimes \alpha_k(b)_{(2)} = (z_i \otimes (\epsilon \otimes \text{id})(\alpha(b))) = z_i \otimes b, \end{aligned}$$

where we used the Sweedler notation for $\alpha_k(b) \in A_k \odot B$ and the leg notation m_{12} for the multiplication. On the other hand by the action equation

$$\begin{aligned} D &= (z_{\bar{i}} \otimes z_i \otimes 1_{\mathcal{M}(B)})(\text{id}_A \otimes \alpha)(\alpha(b)) \\ &= (z_{\bar{i}} \otimes 1_A \otimes 1_{\mathcal{M}(B)})(\text{id}_A \otimes \alpha)((z_i \otimes 1_{\mathcal{M}(B)})\alpha(b)) \in A_{\bar{i}} \odot A_i \odot C_{\bar{i}}, \end{aligned}$$

so that

$$(m_{12} \odot \text{id}_B)(S \odot \text{id}_A \odot \text{id}_B)(D) \in A_i \odot C_{\bar{i}}.$$

This means that $B \subset C_{\bar{i}}$ and the proof of (i) is finished. \square

If B is faithfully and nondegenerately represented on a Hilbert space H , $\mathcal{M}_1(A \otimes B)$ can be viewed as a concrete subalgebra of $B(H \otimes H_\varphi)$. We will often use the following property of $\mathcal{M}_1(A \otimes B)$: for any $y \in \mathcal{M}_1(A \otimes B)$ (so in particular for $y = \alpha(b)$, where $b \in B$ and α is an action of A on B)

$$(2.3) \quad (W^* \otimes I_H)(I_{H_\varphi} \otimes y)(W \otimes I_H) = (\Delta \otimes \text{id}_B)(y).$$

Definition 2.3. Let $\alpha : B \rightarrow \mathcal{M}_1(A \otimes B)$ be an action of a locally compact quantum group A on a C^* -algebra B . A completely bounded map $\gamma : B \rightarrow B$ is said to commute with α if

$$(2.4) \quad (\text{id}_A \otimes \gamma)\alpha = \alpha \circ \gamma.$$

The above definition requires a comment – the formula (2.4) makes sense since one can check that the bounded map $\text{id}_A \otimes \gamma : A \otimes B \rightarrow A \otimes B$ is continuous in the relevant ‘left-strict’ topology and thus extends to a bounded map from $\mathcal{M}_1(A \otimes B)$ to $\mathcal{M}_1(A \otimes B)$.

We are ready to define (a reduced version of) the main object considered in this paper.

Definition 2.4. Let B be a C^* -algebra, faithfully and nondegenerately represented on a Hilbert space H and let $\alpha : B \rightarrow \mathcal{M}_1(A \otimes B)$ be an action of a locally compact quantum group A on B . The (reduced) crossed product of B by the action α is the C^* -subalgebra of $B(H_\varphi \otimes H)$ generated by the products of elements in $\alpha(B)$ and $\hat{A} \otimes I_H$. It will be denoted by $\hat{A} \rtimes_\alpha B$.

If A is commutative, i.e. $A = C_0(G)$ for a locally compact group G , the notion of the crossed product of B by the action α of A coincides with the crossed product of B by the standard action of G induced by α . If A is cocommutative (and the Haar weight is faithful), then A is isomorphic to the reduced C^* -algebra of a locally compact group Γ , the definition of the action of A corresponds to the standard definition of the reduced coaction of Γ and the crossed product defined above coincides with the standard crossed product by α viewed as a coaction ([EKQR]).

As in the classical case we need to know that actually

$$(2.5) \quad \hat{A} \ltimes_{\alpha} B = \text{cl}\{\alpha(B)(\hat{A} \otimes I_H)\}.$$

This can be shown as in Lemma 7.2 of [BS] (see also [Va₂]): for completeness we reproduce the proof below, as in [BS] it is phrased in the language of weak Kac systems. It is enough to show that for all $\omega \in B(H)_*$, $b \in B$ the operator $(\lambda(\omega)^* \otimes I_H)\alpha(b) \in \text{cl}\{\alpha(B)(\hat{A} \otimes I_H)\}$. Note that as A is represented nondegenerately on H_{φ} it is a consequence of the Cohen-Hewitt factorisation theorem ([He]) that there exists $a \in A$ and $\omega' \in B(H)_*$ such that $\omega^* = \omega' a$. Compute then:

$$\begin{aligned} (\lambda(\omega)^* \otimes I_H)\alpha(b) &= ((\omega^* \otimes I_{H_{\varphi}})(W^*) \otimes I_H)\alpha(b) \\ &= (\omega^* \otimes I_{H_{\varphi}} \otimes I_H)((\Delta \otimes \text{id}_B)(\alpha(b))(W^* \otimes I_H)) \\ &= (\omega^* \otimes I_{H_{\varphi}} \otimes I_H)((\text{id}_A \otimes \alpha)(\alpha(b))(W^* \otimes I_H)) \\ &= (\omega' \otimes I_{H_{\varphi}} \otimes I_H)((\text{id}_{B(H)} \otimes \alpha)((a \otimes I_H)\alpha(b))(W^* \otimes I_H)). \end{aligned}$$

As α takes values in $\mathcal{M}_1(A \otimes B)$, the operator $(a \otimes I_H)\alpha(b)$ can be approximated in the norm by finite sums of simple tensors $c_i \otimes d_i$, $c_i \in A$, $d_i \in B$. But

$$\begin{aligned} (\omega' \otimes I_{H_{\varphi}} \otimes I_H)((\text{id}_{B(H)} \otimes \alpha)(c_i \otimes d_i)(W^* \otimes I_H)) \\ = (\omega' c_i \otimes I_{H_{\varphi}} \otimes I_H)((I_{H_{\varphi}} \otimes \alpha(d_i))(W^* \otimes I_H)) = \alpha(d_i)(\lambda(c_i^* \omega'^*) \otimes I_H). \end{aligned}$$

Now the comparison of the formulas above shows that indeed $(\lambda(\omega)^* \otimes I_H)\alpha(b) \in \text{cl}\{\alpha(B)(\hat{A} \otimes I_H)\}$. By density of \hat{A} in \hat{A} and selfadjointness of the latter we deduce that (2.5) holds true.

The definition of $\hat{A} \ltimes_{\alpha} B$ implies that

$$(2.6) \quad \hat{A} \ltimes_{\alpha} B \subset \mathcal{M}_1(K(H_{\varphi}) \otimes B)$$

Indeed, note first that as A is represented nondegenerately on H_{φ} , both $AK(H_{\varphi})$ and $K(H_{\varphi})A$ are dense in $K(H_{\varphi})$ and it follows that $\mathcal{M}_1(A \otimes B) \subset \mathcal{M}_1(K(H_{\varphi}) \otimes B)$. Further a simple computation shows that $\mathcal{M}_1(K(H_{\varphi}) \otimes B)(B(H_{\varphi}) \otimes I_H) \subset \mathcal{M}_1(K(H_{\varphi}) \otimes B)$ and (2.6) is proved.

Remark 2.5. When A is a discrete quantum group and $\alpha : B \rightarrow \mathcal{M}_1(A \otimes B)$ is an action of A , the crossed product $\hat{A} \ltimes_{\alpha} B$ contains a canonical copy of B (recall that \hat{A} is unital, so that $\alpha(B) \subset \hat{A} \ltimes_{\alpha} B$). As ϵ is a vector state on $B(H_{\varphi})$ and (2.6) holds we have a completely positive map $\epsilon \otimes \text{id}_B : \hat{A} \ltimes_{\alpha} B \rightarrow B$, being simply a restriction of the natural map from $\mathcal{M}_1(K(H_{\varphi}) \otimes B)$ to B . Using (2.2) we see that the map $\alpha \circ (\epsilon \otimes \text{id}) : \hat{A} \ltimes_{\alpha} B \rightarrow \alpha(B)$ is a norm one projection so also a conditional expectation onto $\alpha(B)$.

Suppose that $\gamma : B \rightarrow B$ is completely bounded and commutes with α . Then there exists a unique continuous map $\hat{\gamma} : \hat{A} \ltimes_{\alpha} B \rightarrow \hat{A} \ltimes_{\alpha} B$ such that

$$(2.7) \quad \hat{\gamma}(\alpha(b)(x \otimes I_H)) = \alpha(\gamma(b))(x \otimes I_H), \quad b \in B, x \in \hat{A}.$$

The map $\hat{\gamma}$ arises from the natural extension $\tilde{\gamma}$ of the map $\text{id}_{K(H_\varphi)} \otimes \gamma$ to $\mathcal{M}_1(K(H_\varphi) \otimes B)$ (see comments after Definition 2.3). The fact that the resulting map satisfies (2.7) follows from the commutation relation (2.4), property (2.5), ‘left-strict’ continuity of $\text{id}_{K(H_\varphi)} \otimes \gamma$ and appropriate density of $K(H_\varphi) \odot B$ in $\mathcal{M}_1(K(H_\varphi) \otimes B)$. Finally the fact that $\tilde{\gamma}|_{\hat{A} \rtimes_\alpha B}$ has values in $\hat{A} \rtimes_\alpha B$ and the uniqueness of $\hat{\gamma}$ follow from the formula (2.5). It is clear that $\hat{\gamma}$ is completely bounded. Moreover, it is completely positive (nondegenerate, completely contractive) if γ is completely positive (resp. nondegenerate, completely contractive).

In [Va₂] S. Vaes introduced the notion of a universal (full) crossed product (considered also in slightly different guises in [BS] and in [Ng]).

Definition 2.6. Let $\alpha : B \rightarrow \mathcal{M}_1(A \otimes B)$ be an action of A . A pair (X, π) consisting of a unitary corepresentation $X \in \mathcal{M}(A \otimes K(K))$ of A on a Hilbert space K and a nondegenerate $*$ -homomorphism $\pi : B \rightarrow B(K)$ is called a covariant representation of α if for all $b \in B$

$$(\text{id}_A \otimes \pi)(\alpha(b)) = X^*(1_{H_\varphi} \otimes \pi(b))X.$$

A basic example of a covariant representation of α is given by the pair $(W \otimes I_H, \alpha)$, corresponding classically to the left regular representation.

Given an action α as above there exists a (unique up to isomorphism) triple $(\hat{A}_u \rtimes_\alpha B, X_u, \pi_u)$ such that

- (i) $\hat{A}_u \rtimes_\alpha B$ is a C^* -algebra (represented on a Hilbert space H_u);
- (ii) X_u is a unitary in $\mathcal{M}(A \otimes \hat{A}_u \rtimes_\alpha B) \subset \mathcal{M}(A \otimes K(H_u))$, $\pi_u : B \rightarrow \hat{A}_u \rtimes_\alpha B$ is a $*$ -homomorphism and (X_u, π_u) is a covariant representation of α ;
- (iii) the formulas $X = (\text{id}_A \otimes \theta)(X_u)$ and $\pi = \theta\pi_u$ yield a bijective correspondence between covariant representations (X, π) of α and nondegenerate representations θ of $\hat{A}_u \rtimes_\alpha B$.

The algebra $\hat{A}_u \rtimes_\alpha B$ (together with the universal covariant representation (X_u, π_u)) is called the *universal crossed product of B by α* .

It follows from the definitions that there is a canonical $*$ -homomorphism $j_u : \hat{A}_u \rtimes_\alpha B \rightarrow \hat{A} \rtimes_\alpha B$. Proposition 4.4 of [VVe] shows in particular that if A is amenable and α is injective, then j_u is a $*$ -isomorphism. As we are here only interested in the actions of amenable discrete quantum groups, we will discuss only reduced crossed products in the sequel.

3. FACTORISING MAPS ON THE CROSSED PRODUCT BY AN ACTION OF A DISCRETE QUANTUM GROUP

The following theorem is crucial for the main results of the paper formulated in the next section. It shows that certain Schur multiplier type maps on $\hat{A} \rtimes_\alpha B$ can be factorised in a completely positive way via matrices over B . The idea in the case of groups dates back to [Vo] and [SS]. Recall the completely positive maps T_ω on \hat{A} defined in (1.1) and the notion of finitely supported vectors in H_φ introduced at the end of Section 1.

Theorem 3.1. *Let A be a discrete quantum group and let $\xi \in H_\varphi$ be finitely supported, $\xi \in {}_{z_F}H_\varphi$ for some $F \subset \mathcal{I}$ and $\|\xi\| = 1$. Suppose that $B \subset B(H)$ is a nondegenerate (unital) C^* -algebra and $\alpha : B \rightarrow \mathcal{M}_1(A \otimes B)$ is an action of A on B .*

Then there exist nondegenerate (unital) completely positive maps $\Phi_F : \hat{A} \rtimes_\alpha B \rightarrow B(z_F H_\varphi) \otimes B$, $\Psi_\xi : B(z_F H_\varphi) \otimes B \rightarrow \hat{A} \rtimes_\alpha B$ such that

$$(3.1) \quad (\Psi_\xi \circ \Phi_F)(\alpha(b)(x \otimes I_H)) = \alpha(b)(T_{\omega_\xi}(x) \otimes I_H), \quad b \in B, x \in \hat{A}.$$

Moreover if $\gamma : B \rightarrow B$ is a completely bounded map commuting with α and $\hat{\gamma}$ denotes its natural extension to $\hat{A} \rtimes_\alpha B$ given by (2.7) then

$$(3.2) \quad \Phi_F \circ \hat{\gamma} = (\text{id}_{B(z_F H_\varphi)} \otimes \gamma) \circ \Phi_F,$$

and

$$(3.3) \quad \Psi_\xi \circ (\text{id}_{B(z_F H_\varphi)} \otimes \gamma) = \hat{\gamma} \circ \Psi_\xi.$$

Proof. Let $z_F \in Z(A) \subset B(H_\varphi)$ be a finite-rank orthogonal projection and let $\xi \in z_F H_\varphi$, $\|\xi\| = 1$. To simplify the notation we will write in what follows $H_F = z_F H_\varphi$. Let $(e_p)_{p=1}^m$ be an orthonormal basis in H_F . We will often use the fact that in the Dirac notation

$$z_F = \sum_{p=1}^m |e_p\rangle\langle e_p|.$$

Define the map $\Phi_F : \hat{A} \rtimes_\alpha B \rightarrow B(H_F) \otimes B(H)$ via

$$\Phi_F(y) = (z_F \otimes I_H)y(z_F \otimes I_H), \quad y \in \hat{A} \rtimes_\alpha B.$$

Note that Φ_F takes values in $B(H_F) \otimes B$. Indeed, by (2.5) it suffices to show that if $x \in \hat{A}$ and $b \in B$ then $\Phi_F(\alpha(b)(x \otimes I_H)) \in B(H_F) \otimes B$. But

$$(3.4) \quad \begin{aligned} \Phi_F(\alpha(b)(x \otimes I_H)) &= (z_F \otimes I_H)\alpha(b)(x \otimes I_H)(z_F \otimes I_H) \\ &= (z_F \otimes I_H)\alpha(b)(z_F x z_F \otimes I_H) \in B(H_F) \otimes B, \end{aligned}$$

where the second equality and the final inclusion follow from the fact that $\alpha(b) \in \mathcal{M}_1(A \otimes B)$ and $z_F \in Z(A)$. The resulting map Φ_F is clearly completely positive and contractive (unital, if B is unital).

Define a row operator $V_\xi \in B(H_F \otimes H_\varphi; H_\varphi)$ via

$$V_\xi = [\lambda(\omega_{\xi, e_1}) \quad \lambda(\omega_{\xi, e_2}) \quad \cdots \quad \lambda(\omega_{\xi, e_m})]$$

Note that $V_\xi V_\xi^* = I_{H_\varphi}$. Indeed

$$\begin{aligned} V_\xi V_\xi^* &= \sum_{p=1}^m \lambda(\omega_{\xi, e_p}) \lambda(\omega_{\xi, e_p})^* = \sum_{p=1}^m (\omega_{\xi, e_p} \otimes I_{H_\varphi})(W)(\omega_{e_p, \xi} \otimes I_{H_\varphi})(W^*) \\ &= (\langle \xi | \otimes I_{H_\varphi})W(z_F \otimes I_{H_\varphi})W^*(|\xi\rangle \otimes I_{H_\varphi}) = \langle \xi, z_F \xi \rangle I_{H_\varphi} = I_{H_\varphi}. \end{aligned}$$

Let $R_{V_\xi} : B(H_F \otimes H_\varphi \otimes H) \rightarrow B(H_\varphi \otimes H)$ be given by the formula

$$R_{V_\xi}(T) = (V_\xi \otimes I_H)T(V_\xi^* \otimes I_H), \quad T \in B(H_F \otimes H_\varphi \otimes H),$$

and let

$$(3.5) \quad \Psi_\xi = R_{V_\xi} \circ (\text{id}_{B(H_F)} \otimes \alpha).$$

It is then easy to see that if $e_{p,q} = |e_p\rangle\langle e_q|$ ($p, q \in \{1, \dots, m\}$) is a matrix unit in $B(H_F)$, then

$$(3.6) \quad \Psi_\xi(b \otimes e_{p,q}) = (\lambda(\omega_{\xi, e_p}) \otimes I_H)\alpha(b)(\lambda(\omega_{\xi, e_q})^* \otimes I_H), \quad b \in B,$$

so that $\Psi_\xi : B(H_F) \otimes B \rightarrow \hat{A} \rtimes_\alpha B$. It is clearly completely positive, and nondegenerate (unital) as V_ξ is a coisometry.

Recall the definition of the maps T_ω in (1.1). We have for each $x \in \hat{\mathbf{A}}$

$$(3.7) \quad R_{V_\xi}(z_F x z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}}) = T_{\omega_\xi}(x) \otimes I_{\mathbf{H}}.$$

Indeed,

$$\begin{aligned} V_\xi(z_F x z_F \otimes I_{\mathbf{H}_\varphi}) V_\xi^* &= \sum_{p,q=1}^m (\omega_{\xi,e_p} \otimes \text{id}_{B(\mathbf{H}_\varphi)})(W)(\langle e_p, x e_q \rangle I_{\mathbf{H}_\varphi})(\omega_{\xi,e_q} \otimes \text{id}_{B(\mathbf{H}_\varphi)})(W)^* \\ &= (\langle \xi | \otimes I_{\mathbf{H}_\varphi}) W (z_F \otimes I_{\mathbf{H}_\varphi})(x \otimes I_{\mathbf{H}_\varphi})(z_F \otimes I_{\mathbf{H}_\varphi}) W^* (|\xi\rangle \otimes I_{\mathbf{H}_\varphi}) \\ &= (\omega_\xi \otimes \text{id}_{B(\mathbf{H}_\varphi)})(W(x \otimes I_{\mathbf{H}_\varphi}) W^*) = (\text{id}_{B(\mathbf{H}_\varphi)} \otimes \omega_\xi)(\hat{W}^*(I_{\mathbf{H}_\varphi} \otimes x) \hat{W}) \\ &= (\text{id}_{B(\mathbf{H}_\varphi)} \otimes \omega_\xi)(\hat{\Delta}(x)) = T_{\omega_\xi}(x). \end{aligned}$$

Before we establish an explicit formula for the general action of Ψ_ξ , we need to check how the relation (2.1) defining the action property ‘interacts’ with z_F . Let $b \in \mathbf{B}$. Then

$$\begin{aligned} (\text{id}_{B(\mathbf{H}_F)} \otimes \alpha)((z_F \otimes I_{\mathbf{H}})\alpha(b)(z_F \otimes I_{\mathbf{H}})) \\ &= (z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}})(\text{id}_{B(\mathbf{H}_\varphi)} \otimes \alpha)(\alpha(b))(z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}}) \\ &= (z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}})((\Delta \otimes \text{id}_{\mathbf{H}})(\alpha(b)))(z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}}) \end{aligned}$$

Note that the second equality follows easily from the homomorphism property of α , if \mathbf{B} (and therefore also α) is unital. Otherwise one can use a limit argument with the approximate identity of \mathbf{B} . Summarising,

$$(3.8) \quad (\text{id}_{B(\mathbf{H}_F)} \otimes \alpha)((z_F \otimes I_{\mathbf{H}})\alpha(b)(z_F \otimes I_{\mathbf{H}})) \\ = (z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}})((\Delta \otimes \text{id}_{B(\mathbf{H})})(\alpha(b)))(z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}}), \quad b \in \mathbf{B}.$$

Let now $y \in \mathcal{M}_1(\mathbf{A} \otimes \mathbf{B})$. We can view the operator

$$Z := (V_\xi \otimes I_{\mathbf{H}})((z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}})(\Delta \otimes \text{id}_{B(\mathbf{H})})(y))(z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}})$$

as a row of operators in $B(\mathbf{H}_\varphi \otimes \mathbf{H})$, indexed by $p \in \{1, \dots, m\}$. Let us compute its p -th element (recall (2.3)) :

$$\begin{aligned} Z_p &= \sum_{q=1}^m (V_\xi \otimes I_{\mathbf{H}})_q((z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}})(W^* \otimes I_{\mathbf{H}})(I_{\mathbf{H}_\varphi} \otimes y)(W \otimes I_{\mathbf{H}})(z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}}))_{q,p} \\ &= \sum_{q=1}^m (\langle \xi | \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}})(W \otimes I_{\mathbf{H}})(|e_q\rangle \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}}) \\ &\quad (\langle e_q | \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}})(W^* \otimes I_{\mathbf{H}})(I_{\mathbf{H}_\varphi} \otimes y)(W \otimes I_{\mathbf{H}})(|e_p\rangle \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}}). \end{aligned}$$

Moreover,

$$\begin{aligned} Z_p &= (\langle \xi | \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}})(W \otimes I_{\mathbf{H}})(z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}}) \\ &\quad (W^* \otimes I_{\mathbf{H}})(I_{\mathbf{H}_\varphi} \otimes y)(W \otimes I_{\mathbf{H}})(|e_p\rangle \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}}) \\ &= (\langle \xi | \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}})(I_{\mathbf{H}_\varphi} \otimes y)(W \otimes I_{\mathbf{H}})(|e_p\rangle \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}}) \\ &= y((\omega_{\xi,p} \otimes \text{id}_{B(\mathbf{H}_\varphi)})(W) \otimes I_{\mathbf{H}}) = (y(V_\xi \otimes I_{\mathbf{H}}))_p. \end{aligned}$$

Thus we have shown that for all $b \in \mathbf{B}$

$$(3.9) \quad (V_\xi \otimes I_{\mathbf{H}})((z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}})(\Delta \otimes \text{id}_{B(\mathbf{H})})(\alpha(b)))(z_F \otimes I_{\mathbf{H}_\varphi} \otimes I_{\mathbf{H}}) = \alpha(b)(V_\xi \otimes I_{\mathbf{H}}).$$

Now the comparison of the description of the action of Φ_F in (3.4), the definition of Ψ_ξ in (3.5) and the formulas (3.7), (3.8) and (3.9) show that (3.1) holds and the proof of the first part of the theorem is finished.

It remains to check the commutation relations (3.2) and (3.3). The first follows directly from the observation that $\hat{\gamma}$ is just the restriction of $\text{id}_{K(\mathbf{H}_\varphi)} \otimes \gamma$ to $\hat{\mathbf{A}} \rtimes_\alpha \mathbf{B}$. The second is implied by the following consequence of (3.6):

$$\begin{aligned} \Psi(e_{p,q} \otimes \gamma(b)) &= (\lambda(\omega_{\xi,e_p}) \otimes I_{\mathbf{H}}) \alpha(\gamma(b)) (\lambda(\omega_{\xi,e_q})^* \otimes I_{\mathbf{H}}) \\ &= (\lambda(\omega_{\xi,e_p}) \otimes I_{\mathbf{H}}) \hat{\gamma}(\alpha(b)) (\lambda(\omega_{\xi,e_q})^* \otimes I_{\mathbf{H}}) = \hat{\gamma}(\Psi_\xi(e_{p,q} \otimes b)), \end{aligned}$$

where $p, q \in \{1, \dots, m\}, b \in \mathbf{B}$. \square

The assumption of $\|\xi\| = 1$ is used only to assure that Ψ_ξ is (completely) contractive.

4. MAIN THEOREMS

Consider as in [SZ] the following approximation properties for a C^* -algebra \mathbf{B} closely related to properties of the minimal tensor product.

1. Nuclearity, which is equivalent to the CPAP (completely positive approximation property): there exists a net of completely positive contractions $\varphi_\lambda : \mathbf{B} \rightarrow M_{n_\lambda}$ and $\psi_\lambda : M_{n_\lambda} \rightarrow \mathbf{B}$ such that $\psi_\lambda \circ \varphi_\lambda(b) \rightarrow b$ for all $b \in \mathbf{B}$.
2. The CBAP (completely bounded approximation property): there exists a net $(\phi_\lambda : \mathbf{B} \rightarrow \mathbf{B})$ of finite rank maps such that $\phi_\lambda(b) \rightarrow b$ for all $b \in \mathbf{B}$ and $\sup_\lambda \|\phi_\lambda\|_{cb} < \infty$. The smallest possible such supremum is the Haagerup constant $\Lambda(\mathbf{B})$ of \mathbf{B} .
3. The strong OAP (strong operator approximation property): there exists a net $(\phi_\lambda : \mathbf{B} \rightarrow \mathbf{B})$ of finite rank maps such that $(\phi_\lambda \otimes \text{id})(x) \rightarrow x$ for all $x \in \mathbf{B} \otimes B(l^2(\mathbb{N}))$.
4. Exactness, which is equivalent to nuclear embeddability: for every faithful representation $\mathbf{B} \rightarrow B(H)$ there exists a net of completely positive contractions $\varphi_\lambda : \mathbf{B} \rightarrow M_{n_\lambda}$ and $\psi_\lambda : M_{n_\lambda} \rightarrow B(H)$ such that $\psi_\lambda \circ \varphi_\lambda(b) \rightarrow b$ for all $b \in \mathbf{B}$.
5. The OAP (operator approximation property): there exists a net $(\phi_\lambda : \mathbf{B} \rightarrow \mathbf{B})$ of finite rank maps such that $(\phi_\lambda \otimes \text{id})(x) \rightarrow x$ for all $x \in \mathbf{B} \otimes K(l^2(\mathbb{N}))$.

The first four properties are listed in the increasing generality. The OAP neither implies nor follows from exactness, but a C^* -algebra has strong OAP if and only if it is exact and has OAP ([BO]).

The following fact is well known and easy to show (a short proof can be found for example in [SZ]):

Proposition 4.1. *Suppose there exists an approximating net $(\varphi_i : \mathbf{B} \rightarrow \mathbf{C}_i, \psi_i : \mathbf{C}_i \rightarrow \mathbf{B})$ i.e. $\psi_i \circ \varphi_i(b) \rightarrow b$ for all $b \in \mathbf{B}$, where φ_i and ψ_i are contractive and completely positive. If for any of the five approximation properties all \mathbf{C}_i have this property then so does \mathbf{B} , except in case of the CBAP, where \mathbf{B} has the OAP if all \mathbf{C}_i have the CBAP and \mathbf{B} has CBAP if $\sup_i \Lambda(\mathbf{C}_i) < \infty$.*

We also have the following obvious fact:

Proposition 4.2. *Suppose that \mathbf{B} is a C^* -algebra with a C^* -subalgebra \mathbf{C} and there exists a conditional expectation E from \mathbf{B} onto \mathbf{C} . If P is one of the five*

approximation properties listed above and \mathbf{B} has P , then \mathbf{C} also has P (with the Haagerup constant preserved if P is CBAP).

In order to combine Theorem 3.1 with Proposition 4.1 we need to know that one can find the factorisations of the type considered in Theorem 3.1 pointwise converging to the identity on $\hat{\mathbf{A}} \rtimes_{\alpha} \mathbf{B}$. The following result of R. Tomatsu ([To]) can be interpreted as the statement that on an amenable discrete quantum group one can always find ‘approximately invariant finitely supported functions’. Although it is not formulated in [To] exactly in this language, one can easily deduce it from the proof of Theorem 3.9 in that paper. Recall that a discrete quantum group is called *amenable* if its von Neumann algebraic incarnation possesses an invariant mean, i.e. there exists a state $m \in \mathbf{M}^*$ such that

$$m((\omega \overline{\otimes} \text{id}_{\mathbf{M}})(\Delta(x))) = m((\text{id}_{\mathbf{M}} \overline{\otimes} \omega)(\Delta(x))) = \omega(1)m(x), \quad x \in \mathbf{M}, \omega \in \mathbf{M}_*.$$

Theorem 4.3 ([To]). *Let \mathbf{A} be an amenable discrete quantum group. There exists a net of finitely supported vectors $(\xi_i)_{i \in \mathcal{I}}$ such that for each $x \in \hat{\mathbf{A}}$*

$$T_{\omega_{\xi_i}}(x) \xrightarrow{i \in \mathcal{I}} x,$$

where $T_{\omega_{\xi_i}}$ is as in (1.1).

It is also shown in [To] that the existence of a net as above actually characterises amenability of a discrete quantum group. When \mathbf{A} is amenable and commutative, Theorem 4.3 provides the well-known characterisation of amenability via the existence of a suitably normalised Følner net. When \mathbf{A} is cocommutative, so of the form $\widehat{C(G)}$ for some compact group G , it is automatically amenable. The existence of approximations in Theorem 4.3 is in this case a consequence of the Fourier-type isomorphism of \mathbf{H}_{φ} with $L^2(G)$: if ev_e is the state on $C(G)$ given by the evaluation at a neutral element, the map T_{ev_e} is equal to $\text{id}_{C(G)}$ – it remains to observe that ev_e can be approximated in the weak* topology on $C(G)^*$ by measures with continuous densities, so also by measures whose densities are trigonometric polynomials. The latter under the mentioned isomorphism correspond to finitely supported vectors in \mathbf{H}_{φ} . In general Theorem 4.3 can be viewed as asserting the existence of a contractive approximate identity of a specific form for the predual convolution algebra of $\hat{\mathbf{A}}$.

We are ready to state the first of the two main theorems of our paper:

Theorem 4.4. *Suppose that \mathbf{B} is a C^* -algebra equipped with an action of a discrete quantum group \mathbf{A} . Let P be one of the approximation properties listed above. If \mathbf{A} is amenable, then $\hat{\mathbf{A}} \rtimes_{\alpha} \mathbf{B}$ satisfies P if and only if \mathbf{B} satisfies P .*

Proof. Theorem 4.3 together with Theorem 3.1 show that if \mathbf{A} is an amenable discrete quantum group then finitely supported vectors $\xi_i \in \mathbf{H}_{\varphi}$ can be chosen so that the resulting net of multiplier-type maps $\Psi_{\xi_i} \circ \Phi_{F_i}$ constructed in Theorem 3.1 (where F_i denotes the support of ξ_i) provide pointwise norm approximations on $\hat{\mathbf{A}} \rtimes_{\alpha} \mathbf{B}$. Suppose that P is one of the approximation properties and \mathbf{B} has P . As each $z_{F_i} \in B(\mathbf{H}_{\varphi})$ is a finite rank projection, each algebra $B(z_{F_i} \mathbf{H}_{\varphi}) \otimes \mathbf{B}$ also has P and Proposition 4.1 ends the proof of the ‘if’ direction of the theorem.

The ‘only if’ part follows from Proposition 4.2 and Remark 2.5. \square

To formulate the next theorem we need to recall quickly the notion of noncommutative topological entropy due to D. Voiculescu ([Vo], [NS]), in the not necessarily

unital framework. We say that (ϕ, ψ, M_n) is an approximating triple for a C^* -algebra \mathbf{B} if $n \in \mathbb{N}$ and both $\phi : \mathbf{B} \rightarrow M_n$, $\psi : M_n \rightarrow \mathbf{B}$ are completely positive and contractive. We then write $(\phi, \psi, M_n) \in CPA(\mathbf{B})$. Whenever Ω is a finite subset of \mathbf{B} (i.e. $\Omega \in FS(\mathbf{B})$) and $\varepsilon > 0$ the statement $(\phi, \psi, M_n) \in CPA(\mathbf{B}, \Omega, \varepsilon)$ means that $(\phi, \psi, M_n) \in CPA(\mathbf{B})$ and for all $b \in \Omega$

$$\|\psi \circ \phi(b) - b\| < \varepsilon.$$

Nuclearity of \mathbf{B} is equivalent to the fact that for each $\Omega \in FS(\mathbf{B})$ and $\varepsilon > 0$ there exists a triple $(\phi, \psi, M_n) \in CPA(\mathbf{B}, \Omega, \varepsilon)$. For such algebras one can define

$$\text{rcp}(\Omega, \varepsilon) = \min\{n \in \mathbb{N} : \exists \phi : M_n \rightarrow \mathbf{B}, \psi : \mathbf{B} \rightarrow M_n : (\phi, \psi, M_n) \in CPA(\mathbf{B}, \Omega, \varepsilon)\}.$$

Assume now that \mathbf{B} is nuclear and $\gamma : \mathbf{B} \rightarrow \mathbf{B}$ is completely positive and contractive. For any $\Omega \in FS(\mathbf{B})$ and $n \in \mathbb{N}$ let

$$\Omega^{(n)} = \bigcup_{j=0}^{n-1} \gamma^j(\Omega).$$

Then the (Voiculescu) noncommutative topological entropy of γ is given by the formula:

$$\text{ht}(\gamma) = \sup_{\varepsilon > 0, \Omega \in FS(\mathbf{B})} \limsup_{n \rightarrow \infty} \left(\frac{1}{n} \log \text{rcp}(\Omega^{(n)}, \varepsilon) \right).$$

We will also need a ‘dynamical’ version of Proposition 4.1, Lemma 8.1.4 (i) of [NS]. Although it is formulated there only for automorphisms, the same proof works for completely positive contractive maps.

Proposition 4.5. *Let β be a completely positive contractive map on a nuclear C^* -algebra \mathbf{C} , (\mathbf{C}_i) a net of nuclear C^* -algebras together with completely positive contractive maps $\beta_i : \mathbf{C}_i \rightarrow \mathbf{C}_i$, and let $\Phi_i : \mathbf{C} \rightarrow \mathbf{C}_i$, $\Psi_i : \mathbf{C}_i \rightarrow \mathbf{C}$ be two nets of completely positive and contractive equivariant maps (i.e. $\Psi_i \circ \beta_i = \beta \circ \Psi_i$ and $\beta_i \circ \Phi_i = \Phi_i \circ \beta$ for all i) and $c = \lim_i \Psi_i \circ \Phi_i(c)$ for each $c \in \mathbf{C}$. Then $\text{ht}(\beta) \leq \liminf_i \text{ht}(\beta_i)$.*

We are now ready to formulate the theorem on the stability of entropy under taking natural extensions to crossed products by actions of amenable discrete quantum groups. An analogous results for actions of classical groups has been shown in the original paper introducing the noncommutative topological entropy, [Vo].

Theorem 4.6. *Let \mathbf{B} be a nuclear C^* -algebra equipped with an action of an amenable discrete quantum group \mathbf{A} . Suppose that $\gamma : \mathbf{B} \rightarrow \mathbf{B}$ is a completely positive and contractive map commuting with α (i.e. satisfying the condition (2.4)). Denote the canonical extension of γ to $\hat{\mathbf{A}} \rtimes_{\alpha} \mathbf{B}$ by $\hat{\gamma}$. Then $\text{ht} \hat{\gamma} = \text{ht} \gamma$.*

Proof. The proof is similar to that of Theorem 4.4, exploiting additionally the covariance properties of the factorising maps with respect to γ and $\hat{\gamma}$. Theorem 4.3 together with Theorem 3.1 show that finitely supported vectors $\xi_i \in H_{\varphi}$ can be chosen so that the resulting net of multiplier-type maps $\Psi_{\xi_i} \circ \Phi_{F_i}$ constructed in Theorem 3.1 (where F_i denotes the support of ξ_i) provide pointwise norm approximations on $\hat{\mathbf{A}} \rtimes_{\alpha} \mathbf{B}$.

As it is clear that $\text{ht}(\text{id}_{M_n} \otimes \gamma) = \text{ht}(\gamma)$ for all $n \in \mathbb{N}$ we can apply Proposition 4.5 with $\mathbf{C} = \hat{\mathbf{A}} \rtimes_{\alpha} \mathbf{B}$, $\beta = \hat{\gamma}$, $\mathbf{C}_i = B(z_{F_i} H_{\varphi}) \otimes \mathbf{B}$, $\beta_i = \text{id}_{B(z_{F_i} H_{\varphi})} \otimes \gamma$ and the approximating maps $\Psi_i := \Psi_{\xi_i}$, $\Phi_i := \Phi_{F_i}$ to obtain $\text{ht}(\hat{\gamma}) \leq \text{ht}(\gamma)$.

For the other inequality it is enough to invoke the fact that the Voiculescu entropy does not increase under passing to C^* -subalgebras (one may need to use Brown's definition of entropy if the subalgebra is no longer nuclear, but this is not relevant here) and observe that $\hat{\gamma}|_{\alpha(B)} = \alpha \circ \gamma \circ \alpha^{-1}$. \square

Remark 4.7. Theorems 4.4 and 4.6 apply in particular to crossed products by actions of duals of compact groups, i.e. to crossed products by coactions of compact groups ([EKQR]). In fact the analogous results hold for coactions of arbitrary amenable groups, as one can use the Takai-Takesaki duality theorem and apply the standard techniques for crossed products by usual actions. This has been observed for approximation properties in [NiS]. The analogous statement for stability of Voiculescu entropy under natural extensions of maps to crossed products by a coaction of an amenable group can be obtained in a similar manner. The only thing one has to check is that the natural extensions behave well with respect to the Takai-Takesaki duality, but this follows from equivariance properties of dual actions (see appendix A in [EKQR]). We leave the precise formulation of these statements and their proofs to the reader.

5. APPROXIMATION PROPERTIES FOR THE COCYCLE (TWISTED) QUANTUM GROUP ACTIONS

This section contains an extension of the main results proved earlier to the case of cocycle (twisted) crossed products. The extension follows from the stabilisation trick, which states that every cocycle action is stably equivalent to the usual action (i.e. equivalent after tensoring with the identity on the algebra of compact operators). The section is therefore divided into three parts - introduction of basic definitions and properties of cocycle actions of discrete quantum groups and their (twisted) crossed products, a discussion of the stabilisation trick and the statement of the approximation and entropy results for the twisted case.

Cocycle (twisted) quantum group actions and corresponding crossed products. The definition of the cocycle (twisted) action of a locally compact quantum group and of the corresponding crossed product in the von Neumann algebraic framework was given in [VVa]. Here we describe its C^* -algebraic counterpart for the discrete quantum groups. As before we will be only considering faithful left actions.

Definition 5.1. Let B be a C^* -algebra and A a C^* -algebraic discrete quantum group. A cocycle (twisted) action of A on B is a pair (α, U) , where $\alpha : B \rightarrow \mathcal{M}_1(A \otimes B)$ is a faithful nondegenerate $*$ -homomorphism and $U \in \mathcal{M}(A \otimes A \otimes B)$ is a unitary such that for all $b \in B$

$$(5.1) \quad (\text{id}_A \otimes \alpha) \circ \alpha(b) = U((\Delta \otimes \text{id}_B) \circ \alpha(b))U^*$$

and

$$(5.2) \quad (\text{id}_A \otimes \text{id}_A \otimes \alpha)(U)(\Delta \otimes \text{id}_A \otimes \text{id}_B)(U) = (1 \otimes U)(\text{id}_A \otimes \Delta \otimes \text{id}_B)(U).$$

One can show that given a pair (α, U) satisfying the equations (5.1) and (5.2) the $*$ -homomorphism α is faithful if and only if

$$(5.3) \quad (\epsilon \otimes \text{id}_A \otimes \text{id}_B) = \text{Ad}_V,$$

where $V = (\epsilon \otimes \epsilon \otimes \text{id}_{\mathcal{B}})(U)$ is a unitary in \mathcal{B} and $\text{Ad}_V(b) = V^*bV$ for all $b \in \mathcal{B}$. We do not know if this property is in turn equivalent to some natural density conditions on the image of α (see Lemma 2.2 for such statement for the non-twisted actions).

If $\mathbf{A} = C_0(G)$ for a classical locally compact group G then $U \in C_b^{\text{strict}}(G \times G; \mathcal{UM}(\mathcal{B}))$, where $\mathcal{UM}(\mathcal{B})$ denotes the group of unitary elements in $\mathcal{M}(\mathcal{B})$. The conditions (5.1) and (5.2) express then the standard cocycle conditions for a twisted action of G on \mathcal{B} (as given for example in [PR]), with the possible exception of the condition $U(e, t) = U(t, e) = 1_{\mathcal{M}(\mathcal{B})}$ for all $t \in G$, which corresponds precisely to the unitary V featuring in (5.3) to be equal to 1.

If (α, U) is a cocycle action of \mathbf{A} on \mathcal{B} and W denotes the multiplicative unitary of \mathbf{A} , define a unitary $\widetilde{W} \in \mathcal{M}(\mathbf{A} \otimes K(\mathcal{H}_\varphi) \otimes \mathcal{B})$ by the formula

$$\widetilde{W} = (W \otimes 1)U^*.$$

Suppose $\mathcal{B} \subset B(\mathcal{H})$ for some Hilbert space \mathcal{H} and for each $\omega \in B(\mathcal{H}_\varphi)_*$ define $\widetilde{\lambda}(\omega) \in \mathcal{M}(K(\mathcal{H}_\varphi) \otimes \mathcal{B})$ by

$$(5.4) \quad \widetilde{\lambda}(\omega) = (\omega \otimes \text{id}_{B(\mathcal{H}_\varphi)} \otimes \text{id}_{B(\mathcal{H})})(\widetilde{W}).$$

Definition 5.2. Let \mathcal{B} be a C^* -algebra faithfully and nondegenerately represented on a Hilbert space \mathcal{H} and let (α, U) be a cocycle action of a discrete quantum group \mathbf{A} on \mathcal{B} . The (reduced) cocycle (twisted) crossed product of \mathcal{B} by (α, U) is a C^* -subalgebra of $B(\mathcal{H}_\varphi \otimes \mathcal{H})$ generated by the products of elements in $\alpha(\mathcal{B})$ and in $\{\widetilde{\lambda}(\omega) : \omega \in B(\mathcal{H}_\varphi)_*\}$, where $\widetilde{\lambda}(\omega)$ is defined as in (5.4). It will be denoted by $\hat{\mathbf{A}} \rtimes_{\alpha, U} \mathcal{B}$; it is easy to see that $\hat{\mathbf{A}} \rtimes_{\alpha, U} \mathcal{B} \subset M(K(\mathcal{H}_\varphi) \otimes \mathcal{B})$.

We say that a nondegenerate $*$ -homomorphism $\gamma : \mathcal{B} \rightarrow \mathcal{B}$ commutes with (α, U) if

$$(5.5) \quad \alpha \circ \gamma = (\text{id}_{\mathbf{A}} \otimes \gamma) \circ \alpha$$

and

$$(5.6) \quad (\text{id}_{\mathbf{A}} \otimes \text{id}_{\mathbf{A}} \otimes \gamma)(U) = U.$$

Proposition 5.3. *If $\gamma : \mathcal{B} \rightarrow \mathcal{B}$ is a nondegenerate $*$ -homomorphism commuting with the cocycle action (α, U) of a discrete quantum group \mathbf{A} on \mathcal{B} , then there exists a unique $*$ -homomorphism $\hat{\gamma} : \hat{\mathbf{A}} \rtimes_{\alpha, U} \mathcal{B} \rightarrow \hat{\mathbf{A}} \rtimes_{\alpha, U} \mathcal{B}$ such that*

$$(5.7) \quad \hat{\gamma}(\alpha(b)\widetilde{\lambda}(\omega)) = \alpha(\gamma(b))\widetilde{\lambda}(\omega), \quad b \in \mathcal{B}, \omega \in B(\mathcal{H}_\varphi)_*.$$

It is nondegenerate.

Proof. This time, as we assume nondegeneracy, the map $\hat{\gamma}$ arises from the natural extension $\widetilde{\gamma}$ of the $\text{id}_{K(\mathcal{H}_\varphi)} \otimes \gamma$ to $\mathcal{M}(K(\mathcal{H}_\varphi) \otimes \mathcal{B})$. The extension is unital and $*$ -homomorphic. The relations (5.5) and (5.6) and the multiplicativity imply that $\widetilde{\gamma}(\alpha(b)\widetilde{\lambda}(\omega)) = \alpha(\gamma(b))\widetilde{\lambda}(\omega)$ for all $b \in \mathcal{B}$, $\omega \in B(\mathcal{H}_\varphi)_*$. The fact that $\widetilde{\gamma}$ restricts to a nondegenerate map on $\hat{\mathbf{A}} \rtimes_{\alpha, U} \mathcal{B}$ follows easily. \square

When $U = 1$, the twisted notions reduce to the ones introduced before.

Stabilisation trick for C^* -algebraic actions of discrete quantum groups.

The stabilisation trick for classical group actions shows that the crossed products of a C^* -algebra B by a cocycle action of G is equivariantly isomorphic to a crossed product of $B \otimes K$ by a certain usual action of G . A von Neumann algebraic version of the analogous fact for cocycle actions of locally compact quantum groups was proved in [VVa]. Here we explain how to adapt that result to C^* -algebraic actions of discrete quantum groups. The presentation is based on that of [VVa].

Definition 5.4. A cocycle action (α, U) of a discrete quantum group A on a C^* -algebra B is said to be stabilisable by a unitary $X \in M(A \otimes B)$ if

$$(5.8) \quad (1_{M(A)} \otimes X)(\text{id}_A \otimes \alpha)(X) = (\Delta \otimes \text{id}_B)(X)U^*.$$

Lemma 5.5. Suppose that (α, U) is a cocycle action of a discrete quantum group A on a C^* -algebra B stabilisable by a unitary $X \in M(A \otimes B)$. Then the formula

$$(5.9) \quad \beta(b) = X\alpha(b)X^*, \quad b \in B,$$

defines an action of A on B and the map Ad_X mapping $z \mapsto X^*zX$ restricts to a $*$ -isomorphism from $\hat{A} \rtimes_{\beta} B$ onto $\hat{A} \rtimes_{\alpha, U} B$. If a nondegenerate $*$ -homomorphic map $\gamma : B \rightarrow B$ commutes with (α, U) and satisfies $(\text{id}_A \otimes \gamma)(X) = X$, then γ commutes with β and $\text{Ad}_X \circ \hat{\gamma}_{\beta} = \hat{\gamma}_{\alpha} \circ \text{Ad}_X$, where $\hat{\gamma}_{\beta}$ and $\hat{\gamma}_{\alpha}$ denote the respective canonical extensions of γ to $\hat{A} \rtimes_{\beta} B$ and to $\hat{A} \rtimes_{\alpha, U} B$.

Proof. The proof is similar to the one of Proposition 1.8 in [VVa]. Observe first that as $\mathcal{M}_1(A \otimes B) = \bigoplus_{i \in \mathcal{I}} M_{n_i}(B)$ and $\mathcal{M}(A \otimes B) = \prod_{i \in \mathcal{I}} M_{n_i}(B)$ the map β defined by (5.9) actually takes values in $\mathcal{M}_1(A \otimes B)$. Equation (5.8) implies that β satisfies (2.1). As it is faithful, it is an action of A on B .

The formula

$$(\text{id}_A \otimes \text{Ad}_X)(W \otimes 1) = \widetilde{W}(\text{id}_A \otimes \alpha)(X^*)$$

was established in [VVa]. We show now $\text{Ad}_X : \hat{A} \rtimes_{\beta} B \rightarrow \hat{A} \rtimes_{\alpha, U} B$. Due to the continuity it is enough to prove that $\text{Ad}_X((\lambda_{\omega} \otimes 1_{M(B)})\beta(b)) \in \hat{A} \rtimes_{\alpha, U} B$ for all $b \in B$ and a dense set of functionals $\omega \in B(H_{\varphi})_*$. Assume then that $F \subset \subset \mathcal{I}$ (where $A = \bigoplus_{i \in \mathcal{I}} M_{n_i}$) and let $\omega = \omega z_F \in B(H_{\varphi})_*$, $b \in B$. Then

$$\begin{aligned} & \text{Ad}_X((\lambda_{\omega} \otimes 1_{M(B)})\beta(b)) \\ &= (\omega \otimes \text{id}_A \otimes \text{id}_B) \left(\widetilde{W}(\text{id}_A \otimes \alpha)(X^*)(\text{id}_A \otimes \alpha(b))(z_F \otimes 1_{M(B)}) \right) \\ &= (\omega \otimes \text{id}_A \otimes \text{id}_B) \left(\widetilde{W}(\text{id}_A \otimes \alpha)(X^*(z_F \otimes b)) \right). \end{aligned}$$

But $X^*(z_F \otimes b) \in \bigoplus_{i \in F} M_{n_i} \otimes B$ is a linear combination of simple tensors in $A \otimes B$, so that there exist $n \in \mathbb{N}$, $a_1, \dots, a_n \in A$ and $b_1, \dots, b_n \in B$ such that

$$\begin{aligned} & \text{Ad}_X((\lambda_{\omega} \otimes 1_{M(B)})\beta(b)) \\ &= \sum_{i=1}^n (\omega \otimes \text{id}_A \otimes \text{id}_B) \left(\widetilde{W}(a_i \otimes \alpha(b_i)) \right) = \sum_{i=1}^n \tilde{\lambda}(a_i \omega) \alpha(b_i) \in \hat{A} \rtimes_{\alpha, U} B. \end{aligned}$$

The proof that the inverse of Ad_X maps $\hat{A} \rtimes_{\alpha, U} B$ into $\hat{A} \rtimes_{\beta} B$ follows in an analogous way, this time exploiting the adjoint equality

$$(\text{id}_A \otimes \text{Ad}_{X^*})(\widetilde{W}) = (W \otimes 1_{M(B)})(\text{id}_A \otimes \beta)(X)$$

and the fact that Ad_X is a $*$ -homomorphism.

If $\gamma : \mathbf{B} \rightarrow \mathbf{B}$ is a nondegenerate $*$ -homomorphism commuting with (α, U) and $(\text{id}_{\mathbf{A}} \otimes \gamma)(X) = X$, then it is easy to check that γ commutes with β . Let $\omega \in B(\mathbf{H}_\varphi)_*$, $b \in \mathbf{B}$ be as in the proof above. A quick calculation shows that

$$(5.10) \quad \text{Ad}_X \circ \hat{\gamma}_\beta ((\lambda_\omega \otimes 1_{M(\mathbf{B})})\beta(b)) = \sum_{i=1}^n \tilde{\lambda}(a_i \omega) \alpha(b_i),$$

$$(5.11) \quad \hat{\gamma}_\alpha \circ \text{Ad}_X ((\lambda_\omega \otimes 1_{M(\mathbf{B})})\beta(b)) = \sum_{i=1}^k \tilde{\lambda}(c_i \omega) \alpha(\gamma(d_i)),$$

where $X^*(z_F \otimes \gamma(b)) = \sum_{i=1}^n a_i \otimes b_i$ and $X^*(z_F \otimes b) = \sum_{i=1}^k c_i \otimes d_i$. As $(\text{id}_{\mathbf{A}} \otimes \gamma)(X) = X$, we must have $\sum_{i=1}^n a_i \otimes b_i = \sum_{i=1}^k c_i \otimes \gamma(d_i)$, so the expressions (5.10) and (5.11) are equal and the continuous extension arguments end the proof. \square

In the next lemma we use the leg notation for unitaries in the multiplier algebras of tensor products and their ampliations.

Lemma 5.6. *Let (α, U) be a cocycle action of \mathbf{A} on a C^* -algebra \mathbf{B} . Let $V = (\hat{J} \otimes \hat{J})(\Sigma W^* \Sigma)(\hat{J} \otimes \hat{J})$, where \hat{J} is the modular conjugation on \mathbf{H}_φ associated to the Haar state of $\hat{\mathbf{A}}$. Then V is a unitary element of $M(K(\mathbf{H}_\varphi) \otimes \mathbf{A})$ and the cocycle action $(\alpha \otimes \text{id}_{K(\mathbf{H}_\varphi)}, U \otimes 1)$ of \mathbf{A} on $\mathbf{B} \otimes K(\mathbf{H}_\varphi)$ is stabilisable by the unitary $V_{31}^* U_{312}^*$. If a $*$ -homomorphism $\gamma : \mathbf{B} \rightarrow \mathbf{B}$ satisfies (5.6) then $\text{id}_{\mathbf{A}} \otimes \gamma \otimes \text{id}_{K(\mathbf{H}_\varphi)}$ fixes $V_{31}^* U_{312}^*$.*

Proof. Recall that $W \in M(\mathbf{A} \otimes \hat{\mathbf{A}})$. Thus $\Sigma W^* \Sigma \in M(\hat{\mathbf{A}} \otimes \mathbf{A}) \subset M(K(\mathbf{H}_\varphi) \otimes \mathbf{A})$ and to show that $V \in M(K(\mathbf{H}_\varphi) \otimes \mathbf{A})$ it is enough to observe that the adjoint action of the modular conjugation on $B(\mathbf{H}_\varphi)$ leaves \mathbf{A} invariant. The last fact is a consequence of the dual version of Proposition 8.17 of [KV]. As $U_{312}^* \in M(\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{A})$ it follows that $V_{31}^* U_{312}^* \in M(\mathbf{A} \otimes \mathbf{B} \otimes K(\mathbf{H}_\varphi))$. The rest of the argument leading to the first part of the lemma can be conducted exactly as in the von Neumann algebraic case given in Proposition 1.9 of [VVa]. The statement in the second part of the lemma can be checked via a direct computation. \square

Approximation results and equality of entropies for twisted crossed products. The following statements generalise Theorems 4.4 and 4.6 to the case of cocycle crossed products.

Theorem 5.7. *Suppose that \mathbf{B} is a C^* -algebra equipped with a cocycle action (α, U) of a discrete quantum group \mathbf{A} . Let P be one of the approximation properties listed in Section 4. If \mathbf{A} is amenable, then $\hat{\mathbf{A}} \rtimes_{\alpha, U} \mathbf{B}$ satisfies P if and only if \mathbf{B} satisfies P .*

Proof. It is well known that \mathbf{B} has P if and only if $\mathbf{B} \otimes K(\mathbf{H}_\varphi)$ has P . The result therefore follows from Theorem 4.4, Lemma 5.5 and Lemma 5.6. \square

Theorem 5.8. *Let \mathbf{B} be a nuclear C^* -algebra equipped with a cocycle action (α, U) of an amenable discrete quantum group \mathbf{A} . Suppose that $\gamma : \mathbf{B} \rightarrow \mathbf{B}$ is a nondegenerate $*$ -homomorphism commuting with α (i.e. satisfying conditions (5.5) and (5.6)). Denote the canonical extension of γ to $\hat{\mathbf{A}} \rtimes_{\alpha, U} \mathbf{B}$ by $\hat{\gamma}$. Then $\text{ht } \hat{\gamma} = \text{ht } \gamma$.*

Proof. It is easy to see that $\text{ht } \gamma = \text{ht } (\gamma \otimes \text{id}_{K(\mathbf{H}_\varphi)})$ for any completely positive map $\gamma : \mathbf{B} \rightarrow \mathbf{B}$. The result therefore follows from Theorem 4.6, Lemma 5.5 and Lemma 5.6. Note that the last statements of Lemmas 5.5 and 5.6 imply that the stabilisation trick is suitably covariant with respect to $\gamma \otimes \text{id}_{K(\mathbf{H}_\varphi)}$. \square

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